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# Exercises for 'Topics in complex analysis'

(24/09/2025)

**H 3.1 (A convergence criterion)**
Let $D \subset \mathbb{C}$ be a domain and let $f_n : D \to \mathbb{C}$ be a sequence of holomorphic functions that is locally uniformly bounded. Assume that there exists $z_0 \in D$ such that for each $k \in \mathbb{N} \cup \{0\}$ the limit

$$\lim_{n \to +\infty} f_n^{(k)}(z_0)$$

exists. Show that the whole sequence $f_n$ converges locally uniformly to some $f : D \to \mathbb{C}$.

**H 3.2 (From the exam in 2019)**
Denote by $B_1(0) \subset \mathbb{C}$ the open unit disc. Define the family of functions

$$\mathcal{F} = \left\{ f : B_1(0) \to \mathbb{C},\ f \text{ holomorphic},\ \ f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad |a_k| \leq 1 \quad \forall k \in \mathbb{N} \cup \{0\} \right\}.$$

Let $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{F}$ be a sequence. Show that some subsequence of $f_n$ converges locally uniformly to a function $f \in \mathcal{F}$.

**Remark**: The representation $f(z) = \sum_{k=0}^{\infty} a_k z^k$ holds for any holomorphic function $f : B_1(0) \to \mathbb{C}$.

**H 3.3 (An extremal problem in the proof of the Riemann mapping theorem)**
Let $D \subsetneq \mathbb{C}$ be a simply connected domain such that $0 \in D$. Show that there exists a holomorphic function $f : D \to \mathbb{C}$ that solves the extremal problem

$$s_0 = \sup\{|f'(0)| : \text{the function } f : D \to B_1(0) \text{ is holomorphic and injective with } f(0) = 0\}.$$

**Hint**: Consider a sequence of admissible functions $f_n : D \to B_1(0)$ such that $|f_n'(0)| \to s_0$. You may assume without proof that there exists at least one admissible function for the above optimization problem. Indeed, we will prove that later based on the fact that $D$ is simply connected.

**H 3.4 (Local normal vs local uniform convergence of series)**
Let $f_j : U \to \mathbb{C}$ be a sequence of complex-valued functions. The series $\sum_{j=1}^{\infty} f_j$ is called locally normally convergent if for each $z_0 \in U$ there exists $r > 0$ such that

$$\sum_{j=1}^{\infty} \sup_{z \in B_r(z_0)} |f_j(z)| < +\infty.$$

a) Show that if $\sum_{j=1}^{\infty} f_j$ is locally normally convergent then the sequence $g_n := \sum_{j=1}^{n} f_j$ is locally uniformly convergent.

b) Give an example of (not necessarily holomorphic) functions $f_j : \mathbb{C} \to \mathbb{C}$ which shows that the converse is in general false.